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Self-adjoint difference operators and classical solutions to the Stieltjes–Wigert moment problem

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Abstract

The Stieltjes–Wigert polynomials, which correspond to an indeterminate moment problem on the positive half-line, are eigenfunctions of a second order q-difference operator. We consider the orthogonality measures for which the difference operator is symmetric in the corresponding weighted L^2-spaces. Under some additional assumptions these measures are exactly the solutions to the q-Pearson equation. In the case of discrete and absolutely continuous measures the difference operator is essentially self-adjoint, and the corresponding spectral decomposition is given explicitly. In particular, we find an orthogonal set of q-Bessel functions complementing the Stieltjes–Wigert polynomials to an orthogonal basis for $L^2(\mu)$ when μ is a discrete orthogonality measure solving the q-Pearson equation. To obtain the spectral decomposition of the difference operator in case of an absolutely continuous orthogonality measure we use the results from the discrete case combined with direct integral techniques.

MSC: Primary 47B36; secondary 44A60

Keywords: Difference operators; Stieltjes–Wigert polynomials; Spectral analysis; Direct integrals of Hilbert spaces; Self-adjoint operators

1. Introduction

As part of the Askey-scheme [18] of basic hypergeometric orthogonal polynomials, the Stieltjes–Wigert polynomials are eigenfunctions of a second-order q-difference operator. This operator is

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given by

$$\big(Lf\big)(x) = f(xq) - \frac{1}{x}f(x) + \frac{1}{x}f(x/q)$$

or, in a more compact form,

$$L = T_q - x^{-1}(I - T_{q^{-1}}),$$

where T_a denotes the operator defined by $\big(T_a f\big)(x) = f(ax)$ for fixed $a \neq 0$. We always take q as a fixed number in (0, 1). Clearly, L preserves the space of polynomials.

In this paper we consider L as a (possibly) unbounded operator on $L^2(\mu)$, where μ is assumed to be a solution to the Stieltjes–Wigert moment problem, i.e. a positive measure on $[0, \infty)$ such that

$$\int_0^\infty x^n \, d\mu(x) = q^{-\binom{n+1}{2}}, \quad n \geqslant 0. \tag{1.1}$$

Since the Stieltjes–Wigert moment problem is indeterminate, there are infinitely many positive measures to choose from. The operator $(L, \mathcal{P})$ with domain the space $\mathcal{P}$ of polynomials is always symmetric on $L^2(\mu)$. However, the polynomials are only dense in $L^2(\mu)$ when μ is a so-called N-extremal solution to the moment problem, see e.g. [1, Chapter 2]. So instead we consider L with a larger domain $D(L)$ which will be specified in (2.3). Under certain restrictions on $T_{q^{\pm 1}}$, this operator turns out only to be symmetric for a special class of solutions to the moment problem, namely the solutions that satisfy the q-Pearson equation or, in the setup of [10], the solutions that are fixed points of the transformation T defined in [10, Definition 2.4]. Such solutions are also called “classical” in [10]. We give the precise condition that μ has to satisfy in Proposition 2.1.

The question now raises if L can be extended to a self-adjoint operator on $L^2(\mu)$ when μ is a classical solution to the moment problem. We deal with the cases of discrete solutions, respectively, absolutely continuous solutions, in Sections 3 and 4.

In Section 3, where μ is supposed to be discrete, we show that L is unitarily equivalent to a doubly infinite Jacobi operator acting on $\ell^2(\mathbb{Z})$. The theory of unbounded Jacobi operators then leads to the fact that L is essentially self-adjoint. Starting from two explicit eigenfunctions of L constructed in Section 2, the spectrum of L is computed in Theorem 3.3. The spectrum is purely discrete (except for the point 0) and has an unbounded negative part and a bounded positive part. The positive part is simple and each point corresponds to a Stieltjes–Wigert polynomial of fixed degree. The negative part is also simple and each point corresponds now to a q-Bessel function of the second kind. This leads to orthogonality relations for the Stieltjes–Wigert polynomials and for Jackson’s second q-Bessel functions. None of the discrete measures under consideration are canonical solutions in the sense of [1, Definition 3.4.2, p. 115], and hence the space of polynomials has codimension $+\infty$ in the corresponding weighted L^2-spaces. Our analysis leads to an explicit set of orthogonal functions complementing the Stieltjes–Wigert polynomials to a basis for $L^2(\mu)$.

In the case where μ is absolutely continuous, the operator L is again essentially self-adjoint. We show this in Section 4 using direct integrals of Hilbert spaces and the results of Section 3. The spectrum of L has a purely discrete positive part, where each point is of infinite multiplicity and corresponds to a Stieltjes–Wigert polynomials of fixed degree times an arbitrary q-periodic function, i.e. a function f satisfying $f(xq) = f(x)$ for all $x > 0$. In case $\mathrm{supp}(\mu) = [0, \infty)$, the continuous spectrum of L is $(-\infty, 0]$ and each point here is simple. We also give an explicit formula for the spectral measure. The approach in Section 4 should be compared with related ideas of Berg [5].

The indeterminate cases within the Askey-scheme have been classified in [11] and one may ask if a similar construction is possible for other cases as well. For the q-Laguerre polynomials the analysis is already done in [12], where the motivation comes from quantum groups and limit transitions of the big q-Jacobi polynomials. Formal limit results of [12] lead to the results of Section 3, and we note that the methods of Section 4 can be used for the q-Laguerre case as well. See also [9] for the transformation corresponding to the q-Pearson equation. For other cases in the indeterminate part of the Askey-scheme several problems arise, and it is not clear if symmetry of the difference operator for the corresponding orthogonal polynomials has a clear-cut meaning for solutions to the moment problem.

2. Difference operator

2.1. Difference operator

Consider the second order q-difference operator

$$\big(Lf\big)(x) = f(xq) - \frac{1}{x} f(x) + \frac{1}{x} f(x/q). \tag{2.1}$$

The motivation for studying L is the fact that the Stieltjes–Wigert polynomials

$$S_n(x;q) = \frac{1}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} x^k, \quad n = 0, 1, \ldots \tag{2.2}$$

are eigenfunctions of L corresponding to the eigenvalues q^n, see Proposition 2.6 below. Here, we use the notation

$$(q;q)_0 = 1, \quad (q;q)_n = \prod_{k=1}^{n} (1 - q^k), \ \ n = 1, 2, \ldots$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad 0 \leqslant k \leqslant n.$$

Throughout the paper, we assume that $0 < q < 1$ and follow the notation of Gasper and Rahman [15] for basic hypergeometric series.

Recall that the image measure $\tau(\mu)$ of a finite positive measure μ under a measurable map τ is defined by

$$\tau(\mu)(A) = \mu\big(\tau^{-1}(A)\big)$$

for any measurable set A. Recall also that integration with respect to $\tau(\mu)$ is carried out via the rule

$$\int f \, d\tau(\mu) = \int (f \circ \tau) \, d\mu.$$

In what follows we denote by $\tau_a : (0, \infty) \to (0, \infty)$ the map given by $x \mapsto ax$ for fixed $a > 0$.

Writing M for the operator of multiplication by $1/x$, we see that L can be written as

$$L = T_q - M + M \circ T_{q^{-1}}.$$

Our first task is therefore to define and discuss the operators M and $T_{q^{\pm 1}}$ as possibly unbounded operators on $L^2(\mu)$, where μ for the time being is supposed to be any finite positive (Borel) measure on $(0,\infty)$. We define the operator M on the maximal domain

$$D(M) = \left\{ f \in L^2(\mu) \,\Big|\, \int_0^\infty \frac{1}{x^2} |f(x)|^2 \, d\mu(x) < \infty \right\}.$$

As regards the operators $T_{q^{\pm 1}}$, it may happen that one (or both) of them is identically zero on $L^2(\mu)$. This happens if xq (or x/q) never belongs to $\mathrm{supp}(\mu)$ when $x \in \mathrm{supp}(\mu)$ (and hence for example if μ is discrete and supported on $\{tq^{2n} \mid n \in \mathbb{Z}\}$ for some $t > 0$). To avoid this situation we require that $T_{q^{\pm 1}}$, defined on the maximal domains

$$D(T_{q^{\pm 1}}) = \left\{ f \in L^2(\mu) \mid T_{q^{\pm 1}} f \in L^2(\mu) \right\},$$

have trivial kernels, i.e. $\mathrm{Ker}(T_{q^{\pm 1}}) = \{0\}$. For any Borel set $A \subset (0,\infty)$, the indicator function χ_A belongs to $D(T_{q^{\pm 1}})$ since

$$\int_0^\infty |(T_{q^{\pm 1}} \chi_A)(x)|^2 \, d\mu(x) = \mu(q^{\mp 1} A) < \infty.$$

When $\mu(A) > 0$, we have $\chi_A \neq 0$ in $L^2(\mu)$ and the requirement on the kernels therefore implies that $\mu(q^{\mp 1} A) = \tau_{q^{\pm 1}}(\mu)(A) > 0$. In other words, μ is absolutely continuous with respect to $\tau_{q^{\pm 1}}(\mu)$, that is, $\tau_{q^{\pm 1}}$ preserve the support of μ. Note that the domains $D(T_{q^{\pm 1}})$ are dense in $L^2(\mu)$ since the set of finite linear combinations of indicator functions is dense in $L^2(\mu)$.

With the above assumptions in mind we define L as the possibly unbounded operator on $L^2(\mu)$ with domain

$$D(L) = \left\{ f \in L^2(\mu) \mid f \in D(T_q) \cap D(M) \cap D(T_{q^{-1}}),\ T_{q^{-1}} f \in D(M) \right\}. \tag{2.3}$$

Proposition 2.1. *Let μ be a positive measure on $(0,\infty)$ such that*

$$m_n := \int_0^\infty x^n \, d\mu(x) < \infty \quad \text{for } n \geqslant -2.$$

Assume that $T_{q^{\pm 1}} : D(T_{q^{\pm 1}}) \to L^2(\mu)$ have trivial kernels. Then the domain $D(L)$ defined in (2.3) is dense in $L^2(\mu)$ and the operator $(L, D(L))$ is symmetric on $L^2(\mu)$ if and only if the measure $\tau_q(\mu)$ is absolutely continuous with respect to μ and the Radon–Nikodym derivative is given by

$$\frac{d\tau_q(\mu)}{d\mu} = \frac{1}{x} \quad \text{a.e. with respect to } \mu. \tag{2.4}$$

Remark 2.2. When μ is a finite positive measure on $(0,\infty)$ satisfying (2.4), it follows by induction that $\tau_{q^n}(\mu)$ is absolutely continuous with respect to μ for all $n \in \mathbb{Z}$ and

$$\frac{d\tau_{q^n}(\mu)}{d\mu} = \frac{q^{\binom{n}{2}}}{x^n} \quad \text{a.e. with respect to } \mu.$$

This in particular means that μ has moments of all orders and if μ is a probability measure, then

$$\int_0^\infty x^n \, d\mu(x) = q^{-\binom{n+1}{2}} \quad \text{for all } n \in \mathbb{Z}.$$

So the requirement in Proposition 2.1 on the existence of the first two negative moments is actually implied by (2.4). Moreover, we see that μ is uniquely determined by its restriction $\mu|_{(q,1]}$ to the interval $(q, 1]$ (or any other interval of the form $(tq^{k+1}, tq^k]$ for $t > 0$ and $k \in \mathbb{Z}$). See [10, Section 2] for more details.

Proof. Since by assumption $m_{-2} < \infty$, we see that $\chi_A \in D(M)$ for any Borel set $A \subset (0, \infty)$. We have already observed that $\chi_A \in D(T_{q^{\pm 1}})$ and that $T_{q^{-1}}\chi_A = \chi_{qA} \in D(M)$. Hence, all indicator functions are contained in $D(L)$, and finite linear combinations of these functions are dense in $L^2(\mu)$.

Suppose that $f, g \in D(L)$, then

$$\begin{aligned}\langle Lf, g\rangle &= \int_0^\infty \big(Lf\big)(x)\,\overline{g(x)}\,d\mu(x)\\ &= \int_0^\infty \left(f(xq) - \frac{1}{x}f(x) + \frac{1}{x}f(x/q)\right)\overline{g(x)}\,d\mu(x)\\ &= \int_0^\infty f(x)\overline{g(x/q)}\,d\tau_q(\mu)(x) - \int_0^\infty f(x)\frac{\overline{g(x)}}{x}\,d\mu(x)\\ &\quad + \int_0^\infty f(x)\frac{\overline{g(xq)}}{xq}\,d\tau_{q^{-1}}(\mu)(x),\end{aligned}$$

using the fact that each term is integrable. The right-hand side can be written as $\langle f, Lg\rangle$ if and only if

$$\begin{aligned}&\int_0^\infty f(x)\overline{g(qx)}\,d\mu(x) + \int_0^\infty f(x)\frac{\overline{g(x/q)}}{x}\,d\mu(x)\\ &\quad = \int_0^\infty f(x)\,\overline{g(x/q)}\,d\tau_q(\mu)(x) + \int_0^\infty f(x)\,\frac{\overline{g(xq)}}{xq}\,d\tau_{q^{-1}}(\mu)(x).\end{aligned} \tag{2.5}$$

Now, if $\tau_q(\mu)$ and $\tau_{q^{-1}}(\mu)$ are both absolutely continuous with respect to μ and the conditions

$$\frac{d\tau_q(\mu)}{d\mu} = \frac{1}{x} \quad \text{and} \quad \frac{d\tau_{q^{-1}}(\mu)}{d\mu} = xq \ \text{ a.e. with respect to } \mu$$

are met, then (2.5) is satisfied. Since $\tau_{q^{-1}} = \tau_q^{-1}$, these conditions are equivalent and the "if" part of the proposition follows.

Conversely, if $(L, D(L))$ is symmetric, then (2.5) holds for all $f, g \in D(L)$. Take $f = \chi_A$, $g = \chi_B$, then

$$\int_{A\cap q^{-1}B} d\mu(x) + \int_{A\cap qB} \frac{1}{x}\,d\mu(x) = \int_{A\cap qB} d\tau_q(\mu)(x) + \int_{A\cap q^{-1}B} \frac{1}{xq}\,d\tau_{q^{-1}}(\mu)(x).$$

Now take $A \subset (q^{k+1}, q^k]$ for some $k \in \mathbb{Z}$, and set $B = q^{-1}A$ or $A = qB$. This gives $A\cap q^{-1}B = \emptyset$ and therefore

$$\int_A \frac{1}{x}\,d\mu(x) = \tau_q(\mu)(A).$$

Since any Borel set $A \subset (0,\infty)$ can be written as a disjoint union $A = \cup_{k\in\mathbb{Z}} A_k$, where $A_k = A \cap (q^{k+1}, q^k]$, we find that

$$\tau_q(\mu)(A) = \sum_{k\in\mathbb{Z}} \tau_q(\mu)(A_k) = \sum_{k\in\mathbb{Z}} \int_{A_k} \frac{1}{x}\, d\mu(x) = \int_A \frac{1}{x}\, d\mu(x),$$

recalling that $1/x$ is integrable with respect to μ. In particular, $\tau_q(\mu)$ is absolutely continuous with respect to μ and (2.4) is satisfied. □

Remark 2.3. When μ is an N-extremal (or m-canonical) solution to the Stieltjes–Wigert moment problem, then $\tau_{q^{\pm 1}}$ do not preserve the support of μ. See [10, Section 3] for details. So the assumptions on $T_{q^{\pm 1}}$ in Proposition 2.1 exclude canonical solutions of all orders.

In this paper, we shall mainly focus on discrete and absolutely continuous measures and state therefore the following consequence of Proposition 2.1. As for notation, we denote by δ_x the unit mass at the point x.

Corollary 2.4. (i) *Suppose that $t > 0$ and let μ_t be a positive discrete measure of the form*

$$\mu_t = \sum_{k=-\infty}^{\infty} m_t(k)\delta_{tq^k},$$

where $m_t(k) > 0$ for all $k \in \mathbb{Z}$ and $\sum_{k=-\infty}^{\infty} m_t(k) < \infty$. The operator L is symmetric on $L^2(\mu_t)$ if and only if

$$m_t(k+1) = tq^{k+1} m_t(k) \quad \text{for all } k \in \mathbb{Z}. \tag{2.6}$$

(ii) *Let μ be an absolutely continuous measure on $(0,\infty)$ given by a positive density function w satisfying $\int_0^\infty w(x)\,dx < \infty$. Assume that μ and $\tau_{q^{\pm 1}}(\mu)$ have the same support. The operator L is symmetric on $L^2(\mu)$ if and only if*

$$w(xq) = x w(x) \quad \text{for all } x \in (0,\infty). \tag{2.7}$$

Remark 2.5. (i) The condition (2.6) is equivalent to $m_t(k) = t^k q^{\binom{k+1}{2}} m_t(0)$ for $k \in \mathbb{Z}$. If we set $1/m_t(0) = (-tq, -1/t, q; q)_\infty$, it follows by the triple product identity [15, (1.6.1)] that μ_t becomes a probability measure.

(ii) The condition (2.7) is the q-Pearson equation for the Stieltjes–Wigert polynomials, see e.g. [21] and [2]. This equation is for example satisfied by the log-normal density

$$w(x) = \frac{1}{\sqrt{x}} e^{\frac{1}{2}\frac{(\log x)^2}{\log q}}, \quad x > 0$$

and (for fixed $c > 0$) by the infinite products

$$w_c(x) = \frac{x^{c-1}}{(-q^{1-c}x, -q^c/x; q)_\infty}, \quad x > 0.$$

Note also that (2.7) is invariant under multiplication with q-periodic functions, that is, functions which satisfy $f(xq) = f(x)$ for $x > 0$.

In the setting of Proposition 2.1 we find

$$\int_0^\infty |f(xq)|^2\,d\mu(x) = \int_0^\infty \frac{1}{x}|f(x)|^2\,d\mu(x) = q\int_0^\infty \frac{1}{x^2}|f(x/q)|^2\,d\mu(x),$$

showing that L is well-defined on any continuous function f satisfying $f(x) = \mathcal{O}(x^N)$ as $x \to \infty$ and $f(x) = \mathcal{O}(x^{-M})$ as $x \to 0$ for some $N, M \geqslant 0$, cf. Remark 2.2.

2.2. *Eigenfunctions*

The ${}_1\varphi_1$-series with lower parameter equal to zero, say ${}_1\varphi_1\left({a \atop 0}; q, y\right)$, satisfies the second order q-difference equation

$$-\,ay\,f(yq) + (y-q)\,f(y) + q\,f(y/q) = 0. \tag{2.8}$$

This result can be obtained from the second order q-difference equation for the ${}_2\varphi_1$-series [15, Exercise 1.13] by taking a limit.

By looking for solutions of the form $\sum_{k=0}^\infty c_k y^{\lambda+k}$, respectively, $\sum_{k=0}^\infty c_k y^{\lambda-k}$, with $c_0 = 1$, we see that

$${}_1\varphi_1\left({a \atop 0}; q, y\right) \quad \text{and} \quad y^\alpha\,{}_1\varphi_1\left({a \atop 0}; q, \frac{q^2}{y}\right), \quad q^\alpha a = 1 \tag{2.9}$$

both satisfy (2.8).

Proposition 2.6. *The functions defined by*

$$\phi_z(x) = {}_1\varphi_1\left({1/z \atop 0}; q, -xzq\right), \quad \Phi_z(x) = x^{\ln z/\ln q}\,{}_1\varphi_1\left({1/z \atop 0}; q, -\frac{q}{xz}\right)$$

are solutions to the eigenvalue equation $Lf = zf$. Here $\phi_z(x)$ is defined for $x, z \in \mathbb{C}$, where the case $z = 0$ has to be interpreted as the limit

$$\phi_0(x) = {}_0\varphi_1\left({- \atop 0}; q, -xq\right),$$

and $\Phi_z(x)$ is defined for $x \in (0, \infty)$ and $z \in \mathbb{C}\backslash(-\infty, 0]$.

In particular, the Stieltjes–Wigert polynomials are solutions to the eigenvalue equations

$$LS_n(\,\cdot\,; q) = q^n S_n(\,\cdot\,; q), \quad n = 0, 1, \ldots.$$

Remark 2.7. The function

$$\phi_0(x) = \sum_{n=0}^\infty \frac{(-1)^n q^{n^2} x^n}{(q;q)_n}, \quad x \in \mathbb{C}$$

is also known as the entire Rogers–Ramanujan function, since its values at -1 and $-q$ appear in the celebrated identities [15, (2.7.3/4)]

$$\sum_{n=0}^\infty \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q, q^4; q^5)_\infty} \quad \text{and} \quad \sum_{n=0}^\infty \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}.$$

The reader is referred to [3,16] for interesting results about the zeros of ϕ_0, which are all positive and simple.

Proof. The result follows from (2.8) and (2.9) if we replace a by $1/z$ and y by $-xzq$. Since

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} x^k = {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix} ; q, -q^{n+1}x \right),$$

the last assertion follows immediately from (2.2). □

To get hold of the behavior of $\Phi_z(x)$ as $x \downarrow 0$, we need the following result.

Lemma 2.8. *As $x \downarrow 0$, we have*

$${}_0\varphi_1\left(\begin{matrix} - \\ -zq/x \end{matrix} ; q, -\frac{z^2q}{x} \right) \longrightarrow {}_0\varphi_0\left(\begin{matrix} - \\ - \end{matrix} ; q, z \right) = (z;q)_\infty,$$

and the convergence is uniform for z in compact subsets of $\mathbb{C} \setminus (-\infty, 0)$.

Proof. Note that

$${}_0\varphi_1\left(\begin{matrix} - \\ -zq/x \end{matrix} ; q, -\frac{z^2q}{x} \right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q)_n} \frac{z^{2n}}{(x+zq)\cdots(x+zq^n)}$$

for $z \in \mathbb{C} \setminus (-\infty, 0)$ and $x > 0$. The termwise convergence is thus obvious. Let K be a compact subset of $\mathbb{C} \setminus (-\infty, 0)$ and take $\delta > 0$ such that $|z - t| \geqslant \delta$ for all $z \in K$ and $t < 0$. Clearly,

$$|(x+zq)\cdots(x+zq^n)| \geqslant \delta^n q^{\binom{n+1}{2}}$$

and since the right-hand side is independent of $z \in K$ and $x > 0$, we have dominated convergence. □

A limit case of Heine's transformation formula for the ${}_2\varphi_1$-series [18, (0.6.8/9)] tells us that

$${}_1\varphi_1\left(\begin{matrix} 1/z \\ 0 \end{matrix} ; q, -\frac{q}{xz} \right) = (-q/xz;q)_\infty \, {}_0\varphi_1\left(\begin{matrix} - \\ -q/xz \end{matrix} ; q, -\frac{q}{xz^2} \right) \tag{2.10}$$

and according to Lemma 2.8, the ${}_0\varphi_1$-series on the right-hand side converges to $(1/z;q)_\infty$ as $x \downarrow 0$. We follow the convention that in a fraction the part to the right of / is the denominator. So in (2.8), for example, we write $(-q/xz;q)_\infty$ instead of $(-q/(xz);q)_\infty$. The infinite product $(-q/xz;q)_\infty$ does not have a limit as $x \to 0$, but for $x = tq^n$ we have

$$(-q/xz;q)_\infty = (-q^{1-n}/tz;q)_\infty = \frac{(-tz;q)_n(-q/tz;q)_\infty}{(tz)^n q^{\binom{n}{2}}}. \tag{2.11}$$

3. Spectral analysis for the discrete case

In this section, we consider L as an unbounded symmetric operator on the Hilbert space $L^2(\mu_t)$, where μ_t is the discrete measure from Corollary 2.4(i). Throughout the section the parameter $t > 0$ will be fixed.

3.1. $\ell^2(\mathbb{Z})$ setup

Since $L^2(\mu_t)$ essentially is a weighted ℓ^2-space over the integers, we start by defining a unitary operator $U: L^2(\mu_t) \to \ell^2(\mathbb{Z})$ by

$$Uf = \sum_{k=-\infty}^{\infty} f(tq^k)\sqrt{m_t(k)}e_k,$$

where $\{e_k\}_{k\in\mathbb{Z}}$ denotes the standard orthonormal basis for $\ell^2(\mathbb{Z})$. The adjoint of U is given by

$$\big(U^* e_k\big)(tq^r) = \frac{1}{\sqrt{m_t(k)}}\delta_{k,r}$$

and the operator $J = ULU^*$ becomes a doubly infinite Jacobi operator on $\ell^2(\mathbb{Z})$. More precisely, J has the form

$$Je_k = a_k e_{k+1} + b_k e_k + a_{k-1}e_{k-1}, \quad k \in \mathbb{Z}$$

with

$$a_k = \frac{1}{\sqrt{tq^{k+1}}} \quad \text{and} \quad b_k = -\frac{1}{tq^k}.$$

In what follows, we denote by $\mathcal{D}$ the subspace of $\ell^2(\mathbb{Z})$ consisting of finite linear combinations of the basis elements. Clearly, $(J, \mathcal{D})$ is a densely defined symmetric operator on $\ell^2(\mathbb{Z})$. But more importantly, we have the following result.

Theorem 3.1. *The operator $(J, \mathcal{D})$ is essentially self-adjoint.*

By the unitary intertwiner U, the operator $(J, \mathcal{D})$ corresponds to $(L, U^*\mathcal{D}U)$ which is a restriction of the operator $(L, D(L))$ considered in Proposition 2.1. The domain $U^*\mathcal{D}U$ consists of the compactly supported functions in $L^2(\mu)$, and it is straightforward to check that this is a core for the closure of $(L, D(L))$. So by the above theorem, $(L, D(L))$ is essentially self-adjoint in the case $\mu = \mu_t$.

Proof. We employ a theorem of Masson and Repka [22], see also [19, Theorem 4.2.2]. For this we define the operators

$$J^{\pm} := P^{\pm} J\big|_{\mathcal{D}^{\pm}},$$

where P^+ and P^- are the orthogonal projections onto $\mathrm{span}\{e_k \mid k \geqslant 0\}$, respectively, $\mathrm{span}\{e_k \mid k < 0\}$, and

$$\mathcal{D}^+ = \mathcal{D} \cap \mathrm{span}\{e_k \mid k \geqslant 0\}, \quad \mathcal{D}^- = \mathcal{D} \cap \mathrm{span}\{e_k \mid k < 0\}.$$

Note that $J^{\pm}$ are Jacobi operators on $\ell^2(\mathbb{N})$ with finite linear combinations of the basis vectors as domain. The theorem of Masson and Repka states that the deficiency indices of J can be obtained by adding the deficiency indices of J^+ and J^-, see e.g. Akhiezer [1, Chapter 4] or Berezanskiĭ [4, Chapter 7] for more information. The deficiency indices of J^- are $(0, 0)$ since the coefficients a_k and b_k are bounded as $k \to -\infty$. For the deficiency indices of J^+ we observe that $a_k + b_k + a_{k-1}$

is bounded from above for $k \geqslant 0$, and by [1, Addenda and problems to Chapter 1] or [4, Theorem 1.4, p. 505] this implies that J^+ is essentially self-adjoint. Hence, the deficiency indices of J^+ are $(0, 0)$ and we conclude that the deficiency indices of J are also $(0, 0)$. The statement follows. □

The closure of $(J, \mathcal{D})$ thus coincides with the adjoint operator $(J^*, \mathcal{D}^*)$, which is defined on the maximal domain

$$\mathcal{D}^* = \left\{ v \in \ell^2(\mathbb{Z}) : \sum_{k=-\infty}^{\infty} \left| a_k v_{k+1} + b_k v_k + a_{k-1} v_{k-1} \right|^2 < \infty \right\}.$$

3.2. Wronskian and Green function

We now aim at finding the spectrum of the self-adjoint operator $(J^*, \mathcal{D}^*)$. In this connection the functions from Proposition 2.6 become very useful. We set

$$\psi_k(z) = t^{k/2} q^{k(k+1)/4} \phi_z(tq^k),$$

respectively,

$$\Psi_k(z) = t^{k/2} q^{k(k+1)/4} \Phi_z(tq^k) / t^{\ln z / \ln q},$$

and consider the two sequences $\psi(z) = \{\psi_k(z)\}_{k \in \mathbb{Z}}$ and $\Psi(z) = \{\Psi_k(z)\}_{k \in \mathbb{Z}}$. Note that $\psi(z)$ belongs to ℓ^2 as $k \to \infty$ for all $z \in \mathbb{C}$, whereas $\Psi(z)$ belongs to ℓ^2 as $k \to -\infty$ for $z \in \mathbb{C} \setminus \{0\}$. However, except for special values to be determined later on, neither $\psi(z)$ nor $\Psi(z)$ is an element of $\ell^2(\mathbb{Z})$. Since we divide by $t^{\ln z / \ln q}$ in the definition of $\Psi_k(z)$, the sequence $\Psi(z)$ is well-defined for all $z \in \mathbb{C} \setminus \{0\}$.

It follows from Proposition 2.6 that $\psi(z)$ and $\Psi(z)$ are solutions to the eigenvalue equation $Jv = zv$. Their Wronskian, i.e. the sequence defined by

$$[\psi(z), \Psi(z)]_k = a_k \big(\psi_{k+1}(z) \Psi_k(z) - \psi_k(z) \Psi_{k+1}(z) \big), \quad k \in \mathbb{Z}, \tag{3.1}$$

is therefore independent of k.

Lemma 3.2. *The Wronskian of $\psi(z)$ and $\Psi(z)$ is given by*

$$[\psi(z), \Psi(z)] = -z(-tzq, -1/tz, 1/z; q)_\infty.$$

Proof. Inserting the expressions for a_k, $\psi_k(z)$ and $\Psi_k(z)$ in (3.1), we get after a few computations

$$[\psi(z), \Psi(z)]_k = z^k t^k q^{\binom{k+1}{2}} \left\{ {}_1\varphi_1 \left(\begin{matrix} 1/z \\ 0 \end{matrix} ; q, -tzq^{k+2} \right) {}_1\varphi_1 \left(\begin{matrix} 1/z \\ 0 \end{matrix} ; q, -\frac{q^{1-k}}{tz} \right) \right.$$
$$\left. - z\, {}_1\varphi_1 \left(\begin{matrix} 1/z \\ 0 \end{matrix} ; q, -tzq^{k+1} \right) {}_1\varphi_1 \left(\begin{matrix} 1/z \\ 0 \end{matrix} ; q, -\frac{q^{-k}}{tz} \right) \right\}.$$

Since the Wronskian is independent of k, we evaluate the expression by taking the limit $k \to \infty$. Clearly, the ${}_1\varphi_1$-series with argument $-tzq^{k+2}$ (or $-tzq^{k+1}$) converges to 1 as $k \to \infty$. Combining (2.10) with Lemma 2.8 and (2.11), we find that

$$ {}_1\varphi_1 \left(\begin{matrix} 1/z \\ 0 \end{matrix} ; q, -\frac{q^{1-k}}{tz} \right) \sim \frac{(-tz, -q/tz, 1/z; q)_\infty}{(tz)^k q^{\binom{k}{2}}} \quad \text{as } k \to \infty,$$

respectively,

$$ {}_1\varphi_1\left(\begin{matrix}1/z\\0\end{matrix};q,-\frac{q^{-k}}{tz}\right)\sim\frac{(-tz,-q/tz,1/z;q)_\infty}{(tz)^{k+1}q^{\binom{k+1}{2}}}\quad\text{as } k\to\infty, $$

where $\sim$ means that the ratio of the right-hand side and the left-hand side converges to 1 as $k\to\infty$. Therefore,

$$ [\psi(z),\Psi(z)]=\lim_{k\to\infty}\big(q^k-1/t\big)(-tz,-q/tz,1/z;q)_\infty=-z(-tzq,-1/tz,1/z;q)_\infty $$

and the desired result is established. □

With the Wronskian of $\psi(z)$ and $\Psi(z)$ at hand, we define the Green function by

$$ G_z(j,l)=\frac{1}{[\psi(z),\Psi(z)]}\begin{cases}\psi_j(z)\Psi_l(z), & l\leqslant j,\\ \psi_l(z)\Psi_j(z), & l>j.\end{cases} $$

The resolvent of $(J^*,\mathcal{D}^*)$ is closely related to the Green function, see e.g. [19, Section 4.3]. For any sequence $v\in\ell^2(\mathbb{Z})$, we have

$$ \big((J^*-z)^{-1}v\big)_j=\sum_{l=-\infty}^{\infty}G_z(j,l)\,v_l,\quad z\in\mathbb{C}\setminus\mathbb{R}. \tag{3.2} $$

3.3. Spectral decomposition

We denote by E the resolution of the identity corresponding to the self-adjoint operator $(J^*,\mathcal{D}^*)$. From general theory (see e.g. [14, Theorem XII.2.10]) we know that

$$ \begin{aligned}\big\langle E\big((a,b)\big)v,w\big\rangle=\lim_{\delta\downarrow0}\lim_{\varepsilon\downarrow0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}&\langle(J^*-s-i\varepsilon)^{-1}v,w\rangle\\ &-\langle(J^*-s+i\varepsilon)^{-1}v,w\rangle\,ds\end{aligned} \tag{3.3} $$

for $v,w\in\ell^2(\mathbb{Z})$ and because of (3.2), the inner products in the integral can be written as

$$ \big\langle\big(J^*-(s\pm i\varepsilon)\big)^{-1}v,w\big\rangle=\sum_{l\leqslant j}\frac{\psi_j(s\pm i\varepsilon)\Psi_l(s\pm i\varepsilon)}{[\psi(s\pm i\varepsilon),\Psi(s\pm i\varepsilon)]}(v_l\overline{w}_j+v_j\overline{w}_l)(1-\tfrac12\delta_{j,l}). \tag{3.4} $$

Since $\psi_k(z)$ is entire and $\Psi_k(z)$ is analytic in $\mathbb{C}\setminus\{0\}$, it therefore follows that the spectral measure is discrete and supported on the zeros of the Wronskian $[\psi(z),\Psi(z)]$. We can read off these zeros from Lemma 3.2 and get 0, $-q^r/t$ for $r\in\mathbb{Z}$ and q^n for $n\in\mathbb{Z}_+$.

Theorem 3.3. *The spectrum of J^* is given by $\sigma(J^*)=-q^{\mathbb{Z}}/t\cup\{0\}\cup q^{\mathbb{Z}_+}$. The accumulation point 0 does not belong to the point spectrum $\sigma_p(J^*)$.*

Proof. It is only left to prove that 0 does not belong to the point spectrum of J^*. We show that no non-trivial solution to the equation $Jv=0$ belongs to $\ell^2(\mathbb{Z})$. In the end of the proof we use the implication $\phi_0(t)=0\Rightarrow\phi_0(tq)\neq0$, which follows from the fact that the zeros of ϕ_0 are very well separated, see e.g. [10, Section 3].

The space of solutions to the equation $a_k v_{k+1} + b_k v_k + a_{k-1} v_{k-1} = 0$ or, more explicitly,

$$v_{k+1} = \frac{1}{\sqrt{tq^{k-1}}}\, v_k - \sqrt{q}\, v_{k-1}, \quad k \in \mathbb{Z} \tag{3.5}$$

is two-dimensional. We already know one solution, namely $\psi(0)$, which is given by

$$\psi_k(0) = t^{k/2} q^{k(k+1)/4} \phi_0(tq^k), \quad k \in \mathbb{Z}.$$

Clearly $\psi(0)$ belongs to ℓ^2 as $k \to \infty$ but recalling that $\phi_0(tq^{-2n}) \sim (-1)^n t^n q^{-n^2} K(t)$ as $n \to \infty$ for some constant $K(t) > 0$, see e.g. [17], it follows that

$$\psi_{-2n}(0) \sim (-1)^n q^{-n/2} K(t) \quad \text{as } n \to \infty.$$

Therefore, $\psi(0)$ does not belong to $\ell^2(\mathbb{Z})$.

The sequence $\Psi(z)$ is not defined for $z = 0$ so we need to look for other solutions to (3.5). Note that if v_k has the form

$$v_{k+1} = \frac{F_{k+1}}{t^{k/2} q^{k(k-1)/4}},$$

then (3.5) is equivalent to

$$F_{k+1} = F_k - tq^{k-1} F_{k-1}, \quad k \in \mathbb{Z}.$$

With $F_0 = 0$ and $F_1 = 1$ (or, equivalently, $v_0 = 0$ and $v_1 = 1$) we see that F_k, $k = 0, 1, \dots,$ essentially are q-Fibonacci polynomials in t, see e.g. [7]. In particular,

$$F_{k+1} = \sum_{n=0}^{k-1} \begin{bmatrix} k-n \\ n \end{bmatrix}_q (-1)^n q^{n^2} t^n \quad \text{and} \quad F_k \to \phi_0(t) \text{ as } k \to \infty.$$

There are two cases to be considered. (1) When $\phi_0(t) \neq 0$, the solution to (3.5) with $v_0 = 0$ and $v_1 = 1$ does not belong to ℓ^2 as $k \to \infty$. Moreover, since this solution is linearly independent of $\psi(0)$, there are no solutions to (3.5) in $\ell^2(\mathbb{Z})$. (2) In the case $\phi_0(t) = 0$, the solution to (3.5) with $v_0 = 0$ and $v_1 = 1$ is proportional to $\psi(0)$. But since $\phi_0(tq) \neq 0$, the solution to (3.5) with $v_1 = 0$ and $v_2 = 1$ is linearly independent of $\psi(0)$. This solution behaves like $\phi_0(tq)/t^{k/2} q^{k(k-1)/4}$ as $k \to \infty$ and as before we see that no solution to (3.5) belongs to $\ell^2(\mathbb{Z})$. □

3.4. Orthogonality relations

In this section, we determine the spectral measure $E(\{\xi\})$ for ξ in the point spectrum of J^*. Our considerations will lead to explicit orthogonality relations for the Stieltjes–Wigert polynomials and the second q-Bessel functions of Jackson.

Along the way we will need the following auxiliary result.

Lemma 3.4. *For $c \in \mathbb{C}$ and $k, m \in \mathbb{Z}$, we have*

$$(-c)^{m+k}\, {}_1\varphi_1\left(\begin{matrix} -cq^{-m} \\ 0 \end{matrix}; q, q^{1+m+k}\right) = q^{m(m+k)}\, {}_1\varphi_1\left(\begin{matrix} -cq^{-m} \\ 0 \end{matrix}; q, q^{1-m-k}\right). \tag{3.6}$$

Proof. Because of symmetry it suffices to establish the identity for $m+k\geqslant 0$. Applying the transformation [18, (0.6.8/9)], we see that the right-hand side of (3.6) can be written as

$$q^{m(m+k)}\sum_{n=m+k}^{\infty}\frac{(q^{1-m-k+n};q)_\infty}{(q;q)_n}(-c)^n q^{n(n-2m-k)}$$
$$=(-c)^{m+k}\sum_{n=0}^{\infty}\frac{(q^{1+m+k+n};q)_\infty}{(q;q)_n}(-c)^n q^{n(n+k)},$$

which is exactly the left-hand side of (3.6). The special case $c=-1$ can also be obtained by reversing the order of summation. □

From (3.3) and (3.4) it follows that

$$\begin{aligned}\big\langle E\big(\{q^n\}\big)v,w\big\rangle &= \frac{-1}{2\pi i}\oint_{(q^n)}\langle (J^*-s)^{-1}v,w\rangle\,ds\\ &= \frac{-1}{2\pi i}\sum_{l\leqslant j}(v_l\overline{w}_j+v_j\overline{w}_l)(1-\tfrac{1}{2}\delta_{j,l})\oint_{(q^n)}\frac{\psi_j(s)\Psi_l(s)}{[\psi(s),\Psi(s)]}\,ds.\end{aligned}$$

The integral on the right-hand side is given by

$$\frac{-1}{2\pi i}\oint_{(q^n)}\frac{\psi_j(s)\Psi_l(s)}{[\psi(s),\Psi(s)]}\,ds=\psi_j(q^n)\Psi_l(q^n)\operatorname*{Res}_{z=q^n}\frac{1}{[\psi(z),\Psi(z)]}$$

and by Lemma 3.4 (with $c=-1$), we have $\psi_k(q^n)=(-1)^n t^n q^{n^2}\Psi_k(q^n)$. Combining this with the fact that

$$\operatorname*{Res}_{z=q^n}\frac{1}{[\psi(z),\Psi(z)]}=\frac{(-1)^{n+1}t^n q^{n(n+1)}}{(q;q)_n}\frac{1}{(-tq,-1/t,q;q)_\infty},$$

we end up with

$$\big\langle E\big(\{q^n\}\big)v,w\big\rangle=\frac{q^n}{(q;q)_n}\frac{\langle v,\psi(q^n)\rangle\,\langle\psi(q^n),w\rangle}{(-tq,-1/t,q;q)_\infty}.$$

In particular, it follows that

$$\|\psi(q^n)\|^2=\frac{(q;q)_n}{q^n}(-tq,-1/t,q;q)_\infty \quad\text{and}\quad \langle\psi(q^n),\psi(q^m)\rangle=\|\psi(q^n)\|^2\delta_{m,n} \qquad (3.7)$$

if we set $v=w=\psi(q^n)$, respectively $v=w=\psi(q^m)$.

In a similar way as above, one can show that

$$\big\langle E\big(\{-q^r/t\}\big)v,w\big\rangle=\frac{q^r}{(-q/t;q)_r}\frac{\langle v,\psi(-q^r/t)\rangle\,\langle\psi(-q^r/t),w\rangle}{(-t,q,q;q)_\infty}.$$

For by Lemma 3.4, we have $\psi_k(-q^r/t) = (-1)^r q^{r^2} t^{-r} \Psi_k(-q^r/t)$ and

$$\operatorname*{Res}_{z=-q^r/t} \frac{1}{[\psi(z), \Psi(z)]} = \frac{(-1)^{r+1} q^{r(r+1)}}{t^r(-q/t; q)_r} \frac{1}{(-t, q, q; q)_\infty}.$$

It thus follows that

$$\langle \psi(-q^r/t), \psi(-q^s/t) \rangle = \frac{(-q/t; q)_r}{q^r} (-t, q, q; q)_\infty \delta_{r,s}. \tag{3.8}$$

Moreover, we clearly have

$$\langle \psi(q^n), \psi(-q^r/t) \rangle = 0. \tag{3.9}$$

Recall now that the Stieltjes–Wigert polynomials are given by

$$S_n(x; q) = \frac{1}{(q; q)_n} \phi_{q^n}(x)$$

and consider also the functions $M_r^{(t)}(x; q)$ defined by

$$M_r^{(t)}(x; q) = \frac{1}{(q; q)_\infty} \phi_{-q^r/t}(x), \quad r \in \mathbb{Z}.$$

These functions are closely related to the second q-Bessel function [15, Exercise 1.24] defined by

$$\begin{aligned} J_\nu^{(2)}(z; q) &= \frac{(z/2)^\nu}{(q; q)_\infty} {}_1\varphi_1 \left(\begin{matrix} -z^2/4 \\ 0 \end{matrix} ; q, q^{\nu+1} \right) \\ &= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} (z/2)^\nu {}_0\varphi_1 \left(\begin{matrix} - \\ q^{\nu+1} \end{matrix} ; q, -\frac{z^2 q^{\nu+1}}{4} \right). \end{aligned}$$

Indeed, we have $t^{(k+r)/2} M_r^{(t)}(tq^k; q) = q^{r(r+k)/2} J_{k+r}^{(2)}(2\sqrt{t} q^{-r/2}; q)$.

It follows immediately from Proposition 2.6 that

$$L S_n(\cdot\,; q) = q^n S_n(\cdot\,; q) \quad \text{for } n \in \mathbb{Z}_+$$

and

$$L M_r^{(t)}(\cdot\,; q) = -\frac{q^r}{t} M_r^{(t)}(\cdot\,; q) \quad \text{for } r \in \mathbb{Z}.$$

Furthermore, since the spectral decomposition is unique, these eigenfunctions form an orthogonal basis for $L^2(\mu_t)$. We put together the results from (3.7), (3.8) and (3.9) in the following theorem which is a formal limit transition of [12, Theorem 4.1].

Theorem 3.5. *The Stieltjes–Wigert polynomials $S_n(x; q)$, respectively the q-Bessel functions $M_r^{(t)}(x; q)$, are orthogonal in $L^2(\mu_t)$. The orthogonality relations are given by*

$$\frac{1}{(-tq, -1/t, q; q)_\infty} \sum_{k=-\infty}^{\infty} t^k q^{\binom{k+1}{2}} S_n(tq^k; q) S_m(tq^k; q) = \frac{\delta_{m,n}}{q^n (q; q)_n} \tag{3.10}$$

and

$$\frac{1}{(-t;q)_\infty}\sum_{k=-\infty}^{\infty} t^k q^{\binom{k+1}{2}} M_r^{(t)}(tq^k;q) M_s^{(t)}(tq^k;q) = \frac{(-q/t;q)_r}{q^r}\delta_{r,s}. \tag{3.11}$$

Moreover, $S_n(x;q)$ *and* $M_r^{(t)}(x;q)$ *are mutually orthogonal in* $L^2(\mu_t)$, *that is*,

$$\sum_{k=-\infty}^{\infty} t^k q^{\binom{k+1}{2}} S_n(tq^k;q) M_r^{(t)}(tq^k;q) = 0 \quad \text{for all } n, r \tag{3.12}$$

and $\{S_n(x;q)\}_{n\in\mathbb{Z}_+} \cup \{M_r^{(t)}(x;q)\}_{r\in\mathbb{Z}}$ *form an orthogonal basis for* $L^2(\mu_t)$.

Remark 3.6. The orthogonality relation (3.10) is due to Chihara [8], whereas (3.11) is the Hansen–Lommel orthogonality relation for the second q-Bessel function, see [20, Theorem 3.1]. The above theorem contradicts [20, Theorem 3.3], and the flaw in the proof of [20, Theorem 3.3] is contained in [20, Lemma 3.4], where the unbounded operator S as constructed there is not symmetric as claimed.

The statement in (3.12) can also be proved directly in the following way. Use [18, (0.6.8/9)] to write $M_r^{(t)}(x;q)$ as

$$M_r^{(t)}(x;q) = \frac{(xq^{r+1}/t;q)_\infty}{(q;q)_\infty}\,{}_0\varphi_1\left(\begin{matrix} - \\ xq^{r+1}/t \end{matrix};q,-xq\right),$$

so that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} t^k q^{\binom{k+1}{2}} S_n(tq^k;q) M_r^{(t)}(tq^k;q) \\
&= \sum_{k=-\infty}^{\infty} t^k q^{\binom{k+1}{2}} \frac{1}{(q;q)_n}\,{}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix};q,-tq^{k+n+1}\right) \\
&\quad\times \frac{(q^{k+r+1};q)_\infty}{(q;q)_\infty}\,{}_0\varphi_1\left(\begin{matrix} - \\ q^{k+r+1} \end{matrix};q,-tq^{k+1}\right).
\end{aligned}$$

Because of absolute convergence we can interchange the order of summation to get

$$\begin{aligned}
&\frac{1}{(q;q)_n}\sum_{m=0}^{n} \frac{(q^{-n};q)_m}{(q;q)_m} t^m q^{\binom{m}{2}+m(n+1)} \frac{1}{(q;q)_\infty}\sum_{l=0}^{\infty}\frac{(-1)^l q^{l^2} t^l}{(q;q)_l} \\
&\quad\times \sum_{k=-\infty}^{\infty} (q^{k+r+l+1};q)_\infty t^k q^{\binom{k}{2}+k(m+l+1)}.
\end{aligned}$$

The inner sum (over k) reduces to

$$\begin{aligned}
\sum_{k=-r-l}^{\infty} (q^{k+r+l+1};q)_\infty t^k q^{\binom{k}{2}+k(m+l+1)} &= \frac{(q;q)_\infty q^{\binom{r+l}{2}}}{t^{r+l} q^{(r+l)(m+l)}} \sum_{k=0}^{\infty} \frac{t^k q^{\binom{k}{2}+k(m+1-r)}}{(q;q)_k} \\
&= \frac{(-tq^{m+1-r},q;q)_\infty q^{\binom{r}{2}+\binom{l}{2}}}{q^{l^2+m(r+l)} t^{r+l}}
\end{aligned}$$

and the sum over l then becomes

$$\sum_{l=0}^{\infty} \frac{(-1)^l q^{\binom{l}{2}-lm}}{(q;q)_l} = (q^{-m};q)_\infty.$$

Since $(q^{-m};q)_\infty = 0$ for $m \geqslant 0$, the relation (3.12) is established.

Remark 3.7. Using the explicit expression for $M_r^{(t)}(x;q)$ and Lemma 3.4, we see that $|M_r^{(t)}(tq^k;q)|$ is bounded by some constant, say $M(r,t)$, for all $k \in \mathbb{Z}$ provided $t < q^r$. By the construction of Berg [6] it thus follows from Theorem 3.5 that the measure

$$\nu_{s,t} = \frac{1}{(-tq,-1/t,q;q)_\infty} \sum_{k=-\infty}^{\infty} t^k q^{\binom{k+1}{2}} \left(1 + \frac{s}{M(r,t)} M_r^{(t)}(tq^k;q)\right) \delta_{tq^k}$$

is a solution to the Stieltjes–Wigert moment problem for all $|s| \leqslant 1$ and $t < q^r$.

4. Spectral analysis for the continuous case

We now work on the Hilbert space $L^2(\mu)$, where μ is the absolutely continuous measure from Corollary 2.4(ii). The density of μ, which will be denoted w, thus satisfies the functional equation

$$w(xq) = xw(x), \quad x > 0. \tag{4.1}$$

We remind the reader that a function g is called q-periodic if $g(xq) = g(x)$ for all $x > 0$.

4.1. Direct integral decomposition

Consider the Hilbert space $\ell^2(\mathbb{Z})$ equipped with its standard orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$. For a compactly supported measurable function f on $(0,\infty)$ we define

$$\begin{aligned}(q,1] \ni t \mapsto (If)(t) &= \sum_{k=-\infty}^{\infty} f(tq^k) q^{k/2} \sqrt{w(tq^k)}\, e_k \\ &= \sqrt{w(t)} \sum_{k=-\infty}^{\infty} f(tq^k) t^{k/2} q^{k(k+1)/4}\, e_k \in \ell^2(\mathbb{Z}).\end{aligned} \tag{4.2}$$

Clearly, $(I(gf))(t) = g(t)(If)(t)$ whenever g is a q-periodic function.

Proposition 4.1. *The operator I defined in* (4.2) *extends to a unitary isomorphism*

$$I : L^2(\mu) \to \int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$$

with $\Omega = (q,1] \cap \operatorname{supp}(\mu)$.

Remark 4.2. The direct integral Hilbert space $\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$ consists of all measurable functions $f : \Omega \to \ell^2(\mathbb{Z})$ with $\int_\Omega \|f(t)\|^2_{\ell^2(\mathbb{Z})}\, dt < \infty$. The term measurable means that $t \mapsto$

$\langle f(t), e_k\rangle_{\ell^2(\mathbb{Z})}$ is measurable for all $k \in \mathbb{Z}$. In particular, the constant vector fields $t \mapsto e_j$ are measurable. The inner product on $\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$ is given by

$$\langle f, g\rangle_{\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt} = \int_\Omega \langle f(t), g(t)\rangle_{\ell^2(\mathbb{Z})}\, dt$$

and we have $\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt \cong L^2(\Omega) \otimes \ell^2(\mathbb{Z})$ as Hilbert spaces. The space of all $t \mapsto g(t)e_j$, g bounded measurable function on Ω, is therefore dense in $\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$. Notice that $t \mapsto (If)(t)$ as defined in (4.2) is measurable. See e.g. [13, Part II, Chapter 1] for more information.

Proof. For f, g compactly supported functions in $L^2(\mu)$, we have

$$\begin{aligned}\langle If, Ig\rangle_{\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt} &= \int_\Omega \langle (If)(t), (Ig)(t)\rangle_{\ell^2(\mathbb{Z})}\, dt = \int_\Omega \sum_{k=-\infty}^{\infty} f(tq^k)\overline{g(tq^k)}q^k w(tq^k)\, dt \\ &= \sum_{k=-\infty}^{\infty} \int_q^1 f(tq^k)\overline{g(tq^k)}q^k w(tq^k)\, dt = \sum_{k=-\infty}^{\infty} \int_{q^{k+1}}^{q^k} f(x)\overline{g(x)}w(x)\, dx \\ &= \int_0^\infty f(x)\overline{g(x)}w(x)\, dx = \langle f, g\rangle_{L^2(\mu)},\end{aligned}$$

where interchanging summation and integration is allowed since f, g being compactly supported implies that the sum is finite. Moreover, we can switch from $\int_\Omega$ to $\int_q^1$ since w satisfies the functional equation (4.1).

Recalling that the compactly supported measurable functions are dense in $L^2(\mu)$, the operator I from (4.2) extends to an isometry $I : L^2(\mu) \to \int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$. Since the image of I contains any element of the form $t \mapsto h(t)e_k$, h bounded measurable function on Ω, and these elements are dense in $\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$, we conclude that $I: L^2(\mu) \to \int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$ is surjective and thus unitary. □

The adjoint of the unitary operator I is given explicitly by

$$I^*\left(t \mapsto \sum_{k=-\infty}^{\infty} h_k(t)\, e_k\right)(x) = \sum_{k=-\infty}^{\infty} \chi_{(q^{k+1},q^k]}(x)\frac{h_k(xq^{-k})}{q^{k/2}\sqrt{w(x)}}, \tag{4.3}$$

where χ_A denotes the indicator function of the set A. The right-hand side of (4.3) only makes sense when $w(x) > 0$, but there is no need to specify the value of a function in $L^2(\mu)$ at points where $w(x) = 0$. Formally calculating $I\phi_z$, with ϕ_z the eigenfunction of L from Proposition 2.6, gives

$$(I\phi_z)(t) = \sqrt{w(t)} \sum_{k=-\infty}^{\infty} \phi_z(tq^k)t^{k/2}q^{k(k+1)/4}\, e_k = \sqrt{w(t)}\, \psi(z; t),$$

with $\psi(z; t)$ the formal, i.e. in general not contained in $\ell^2(\mathbb{Z})$, eigenvectors of J_t as in Section 3.2. Conversely, by (4.3) we have for any function f on Ω that

$$I^*\left(t \mapsto f(t) \sum_{k=-\infty}^{\infty} \phi_z(tq^k)t^{k/2}q^{k(k+1)/4}e_k\right) = \mathrm{Per}(f/\sqrt{w})\, \phi_z,$$

where Per maps a function on Ω to a q-periodic function on $\mathrm{supp}(\mu)$ such that they are equal on Ω, explicitly

$$\mathrm{Per}(f)(x) = \sum_{k=-\infty}^{\infty} \chi_{(q^{k+1},q^k]}(x) f(xq^{-k}). \tag{4.4}$$

Recall from Section 3.1 the unbounded symmetric operator $(J_t, \mathcal{D})$ on $\ell^2(\mathbb{Z})$ defined by

$$J_t e_k = a_k(t)e_{k+1} + b_k(t)e_k + a_{k-1}(t)e_{k-1}, \quad k \in \mathbb{Z}$$

with

$$a_k(t) = \frac{1}{\sqrt{tq^{k+1}}} \quad \text{and} \quad b_k(t) = -\frac{1}{tq^k}.$$

Note that a_k and b_k are bounded continuous functions of $t \in (q, 1]$ for fixed $k \in \mathbb{Z}$. It follows from Theorem 3.1 that $(J_t, \mathcal{D})$ is essentially self-adjoint, and we denote by $(J_t^*, \mathrm{dom}(J_t^*))$ its unique self-adjoint extension.

Let $L^2(\Omega) \otimes \mathcal{D}$ be the (algebraic) tensor product of the space $L^2(\Omega)$ and the space $\mathcal{D}$ of finite linear combinations of the basis vectors. By Remark 4.2 this tensor product is dense in $\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$ since it contains $B(\Omega) \otimes \mathcal{D}$, with $B(\Omega)$ the space of bounded measurable functions on Ω. Observe that for $h \otimes v \in L^2(\Omega) \otimes \mathcal{D}$, the field $t \mapsto h(t) J_t v$ is measurable because the inner product

$$t \mapsto \langle h(t) J_t v, e_k \rangle = h(t) \langle v, J_t e_k \rangle = h(t) \big(a_k(t) \langle v, e_{k+1} \rangle + b_k(t) \langle v, e_k \rangle + a_{k-1}(t) \langle v, e_{k-1} \rangle \big)$$

is measurable for any $k \in \mathbb{Z}$. Moreover, this inner product is only non-zero for finitely many values of k, so the vector field $t \mapsto h(t) J_t v$ is an element of $\int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt$. We now define $\int_\Omega^\oplus J_t\, dt$ as the operator with domain $L^2(\Omega) \otimes \mathcal{D}$ mapping the element $h \otimes v$ considered as the vector field $t \mapsto h(t)v$ to $t \mapsto h(t) J_t v$. Note that $h \otimes v$ is identified with $f \otimes v$ whenever $f = h$ a.e. in Ω.

Proposition 4.3. *Consider L as an unbounded operator with domain the compactly supported functions in $L^2(\mu)$. Then I intertwines L with $J = \int_\Omega^\oplus J_t\, dt$.*

Proof. For f compactly supported, take $N, M \in \mathbb{Z}$ such that $\mathrm{supp}(f) \subset (q^{N+1}, q^M]$ and identify

$$(If)(t) = \sum_{k=N}^{M} f(tq^k) q^{k/2} \sqrt{w(tq^k)} e_k$$

with $\sum_{k=N}^M h_k \otimes e_k \in L^2(\Omega) \otimes \mathcal{D}$, where $h_k(t) = f(tq^k) q^{k/2} \sqrt{w(tq^k)}$. Since

$$\int_\Omega |h_k(t)|^2\, dt = \int_{q^{k+1}}^{q^k} |f(x)|^2 w(x)\, dx < \infty,$$

we have indeed $h_k \in L^2(\Omega)$. So I maps the domain of L into $L^2(\Omega) \otimes \mathcal{D}$. Conversely, I^* of an element $h \otimes e_k \in L^2(\Omega) \otimes \mathcal{D}$ gives by (4.3) a compactly supported function on $(0, \infty)$ and

$$\int_0^\infty |I^*(h \otimes e_k)(x)|^2 w(x)\, dx = \int_\Omega |h(t)|^2\, dt < \infty.$$

The intertwining property is a straightforward calculation. For $f \in \mathrm{dom}(L)$ and fixed $t \in \Omega$, we have

$$\begin{aligned} I(Lf)(t) &= \sqrt{w(t)} \sum_{k=-\infty}^{\infty} \left(f(tq^{k+1}) - \frac{1}{tq^k} f(tq^k) + \frac{1}{tq^k} f(tq^{k-1}) \right) t^{k/2} q^{k(k+1)/4} e_k \\ &= \sqrt{w(t)} \sum_{k=-\infty}^{\infty} f(tq^k) \left(\frac{1}{\sqrt{tq^k}}\, e_{k-1} - \frac{1}{tq^k}\, e_k + \frac{1}{\sqrt{tq^{k+1}}}\, e_{k+1} \right) t^{k/2} q^{k(k+1)/4} \\ &= J_t(If)(t). \end{aligned}$$

Note that the infinite sums only contain a finite number of non-zero terms, so that all rearrangements are valid. □

Since the operator L from Proposition 4.3 is symmetric and commutes with complex conjugation, it has a self-adjoint extension. We aim at finding its adjoint for which we want to give a direct integral representation. Because of Proposition 4.3 and the fact that each $(J_t^*, \mathrm{dom}(J_t^*))$ is self-adjoint we consider the operator $J^* = \int_\Omega^\oplus J_t^* dt$. The next paragraph justifies this notation.

According to [23, Definition p. 283] we need to check that the field of operators $t \mapsto (J_t^* + i)^{-1}$ is measurable, i.e. that $t \mapsto \langle (J_t^* + i)^{-1} e_k, e_l \rangle_{\ell^2(\mathbb{Z})}$ is measurable for all $k, l \in \mathbb{Z}$. By the functional calculus for J_t^* established in Section 3, we have

$$\langle (J_t^* + i)^{-1} e_k, e_l \rangle_{\ell^2(\mathbb{Z})} = \int_{\mathbb{R}} \frac{1}{\lambda + i}\, dE^t_{e_k, e_l}(\lambda),$$

where the right-hand side can be written as

$$\sum_{n=0}^{\infty} \frac{1}{q^n + i} \frac{\langle e_k, \psi(q^n; t) \rangle \langle \psi(q^n; t), e_l \rangle}{\|\psi(q^n; t)\|^2} + \sum_{r=-\infty}^{\infty} \frac{1}{i - q^r/t} \frac{\langle e_k, \psi(-q^r/t; t) \rangle \langle \psi(-q^r/t; t), e_l \rangle}{\|\phi(-q^r/t; t)\|^2}.$$

The desired measurability hence follows. Now define

$$\mathrm{dom}(J^*) = \left\{ t \mapsto u(t) \in \int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt \,\middle|\, u(t) \in \mathrm{dom}(J_t^*) \text{ a.e.}, \int_\Omega \|J_t^* u(t)\|^2\, dt < \infty \right\},$$

$$J^* = \int_\Omega^\oplus J_t^*\, dt : \mathrm{dom}(J^*) \ni \big(t \mapsto u(t)\big) \longmapsto \big(t \mapsto J_t^* u(t)\big).$$

By [23, Theorem XIII.85, p. 284] the operator $J^* = \int_\Omega^\oplus J_t^* dt$ is the adjoint of J and J^* is self-adjoint. Moreover, the functional calculus is given by

$$f(J^*) = f\left(\int_\Omega^\oplus J_t^*\, dt \right) = \int_\Omega^\oplus f(J_t^*)\, dt \tag{4.5}$$

for any bounded measurable function f on $\mathbb{R}$.

Proposition 4.4. *The adjoint operator* $(L^*, \mathrm{dom}(L^*))$ *is intertwined with* $(J^*, \mathrm{dom}(J^*))$ *by the unitary isomorphism* I.

As an immediate consequence, we have

Corollary 4.5. $(L^*, \mathrm{dom}(L^*))$ *is the unique self-adjoint extension of* $(L, \mathrm{dom}(L))$, *and for any bounded Borel function f on* $\mathbb{R}$ *the functional calculus is given by*

$$f(L^*) = I^* \int_\Omega^\oplus f(J_t^*)\, dt\, I.$$

Proof of Proposition 4.4. The domain of L^* consists of all functions $g \in L^2(\mu)$ such that

$$f \mapsto \langle Lf, g\rangle_{L^2(\mu)} = \int_\Omega \langle I(Lf)(t), (Ig)(t)\rangle_{\ell^2(\mathbb{Z})}\, dt = \int_\Omega \langle J_t(If)(t), (Ig)(t)\rangle_{\ell^2(\mathbb{Z})}\, dt$$

defines a continuous linear functional on $\mathrm{dom}(L)$. We have used Proposition 4.3 to replace $I(Lf)$ with $J_t(If)$ in the inner product on the right-hand side. So for $g \in \mathrm{dom}(L^*)$ there exists a constant $C = C(g) > 0$ such that

$$\begin{aligned} |\langle Lf, g\rangle_{L^2(\mu)}| &= \left| \int_\Omega \langle J_t(If)(t), (Ig)(t)\rangle_{\ell^2(\mathbb{Z})}\, dt \right| \leqslant C \|f\|_{L^2(\mu)} \\ &= C \left(\int_\Omega \|If(t)\|^2_{\ell^2(\mathbb{Z})}\, dt \right)^{1/2} \end{aligned} \tag{4.6}$$

for all $f \in \mathrm{dom}(L)$. Since f is compactly supported, the inner product $\langle J_t(If)(t), (Ig)(t)\rangle_{\ell^2(\mathbb{Z})}$ is a finite sum and hence equal to $\langle If(t), J_t^*(Ig)(t)\rangle_{\ell^2(\mathbb{Z})}$. Therefore, (4.6) can be rewritten as

$$\left| \int_\Omega \langle If(t), J_t^*(Ig)(t)\rangle_{\ell^2(\mathbb{Z})}\, dt \right| \leqslant C \left(\int_\Omega \|If(t)\|^2_{\ell^2(\mathbb{Z})}\, dt \right)^{1/2}. \tag{4.7}$$

Now we can show that the vector field $t \mapsto Ig(t)$ belongs to $\mathrm{dom}(\int_\Omega^\oplus J_t^*\, dt)$ whenever $g \in \mathrm{dom}(L^*)$. First, by taking $f = I^*(1 \otimes e_k) \in \mathrm{dom}(L)$ we see that

$$t \mapsto \langle e_k, J_t^*(Ig)(t)\rangle_{\ell^2(\mathbb{Z})} = \langle If(t), J_t^*(Ig)(t)\rangle_{\ell^2(\mathbb{Z})} = \langle J_t(If)(t), Ig(t)\rangle_{\ell^2(\mathbb{Z})}$$

is measurable and square integrable on Ω for any $k \in \mathbb{Z}$, since

$$\begin{aligned} \langle J_t(If)(t), Ig(t)\rangle_{\ell^2(\mathbb{Z})} &= a_k(t)\langle e_{k+1}, Ig(t)\rangle_{\ell^2(\mathbb{Z})} + b_k(t)\langle e_k, Ig(t)\rangle_{\ell^2(\mathbb{Z})} \\ &\quad + a_{k-1}(t)\langle e_{k-1}, Ig(t)\rangle_{\ell^2(\mathbb{Z})}. \end{aligned}$$

Then apply (4.7) with $If(t) = \sum_{k=-N}^N \langle J_t^*(Ig)(t), e_k\rangle_{\ell^2(\mathbb{Z})} e_k$ to get

$$\sum_{k=-N}^N \int_\Omega |\langle J_t^*(Ig)(t), e_k\rangle_{\ell^2(\mathbb{Z})}|^2\, dt \leqslant C \left(\int_\Omega \sum_{k=-N}^N |\langle J_t^*(Ig)(t), e_k\rangle_{\ell^2(\mathbb{Z})}|^2\, dt \right)^{1/2}$$

or

$$\sum_{k=-N}^N \int_\Omega |\langle J_t^*(Ig)(t), e_k\rangle_{\ell^2(\mathbb{Z})}|^2\, dt \leqslant C^2.$$

Since C is independent of N, this is also valid for $N \to \infty$. In particular, it follows that

$$\sum_{k=-\infty}^{\infty} |\langle J_t^*(Ig)(t), e_k\rangle_{\ell^2(\mathbb{Z})}|^2 < \infty \quad \text{a.e.}$$

so that $t \mapsto J_t^*(Ig)(t)$ is a measurable square integrable vector field for which $Ig(t) \in \mathrm{dom}(J_t^*)$ a.e. This proves that $I\mathrm{dom}(L^*) \subset \mathrm{dom}(\int_\Omega^\oplus J_t^*\, dt)$ and IL^* is the restriction of $J^*I = \int_\Omega^\oplus J_t^*\, dt\, I$.

For the converse inclusion take $g \in I^*\mathrm{dom}(J^*)$ and observe that for any $f \in \mathrm{dom}(L)$,

$$\begin{aligned}|\langle Lf, g\rangle_{L^2(\mu)}| &= \left|\int_\Omega \langle J_t(If)(t), Ig(t)\rangle_{\ell^2(\mathbb{Z})}\, dt\right| = \left|\int_\Omega \langle If(t), J_t^*(Ig)(t)\rangle_{\ell^2(\mathbb{Z})}\, dt\right| \\ &\leqslant \left(\int_\Omega \|If(t)\|^2_{\ell^2(\mathbb{Z})}\, dt\right)^{1/2} \left(\int_\Omega \|J_t^*(Ig)(t)\|^2_{\ell^2(\mathbb{Z})}\, dt\right)^{1/2} = C\|f\|_{L^2(\mu)}.\end{aligned}$$

In other words, $f \mapsto \langle Lf, g\rangle_{L^2(\mu)}$ defines a continuous linear functional on $\mathrm{dom}(L)$ and it follows that $I^*\mathrm{dom}(J^*) \subset \mathrm{dom}(L^*)$. □

4.2. *Spectral decomposition for L^**

We start this section by presenting the spectrum of L^*.

Theorem 4.6. *The spectrum of the self-adjoint operator $(L^*, \mathrm{dom}(L^*))$ consists of point spectrum $q^{\mathbb{Z}_+}$, each point having infinite multiplicity, and continuous spectrum $\overline{\cup_{l\in\mathbb{Z}}\tilde{\Omega}_l}$, where $\tilde{\Omega}_l = \{-q^l/t \mid t \in \Omega\}$. In particular, we have $\sigma(L^*) = (-\infty, 0] \cup q^{\mathbb{Z}_+}$ when $\Omega = (q, 1]$.*

Proof. The theorem follows from [23, Theorem XIII.85] and Proposition 4.4. We only need to consider the point 0 which is in the closure of $q^{\mathbb{Z}_+}$ and in the closure of $\cup_{l\in\mathbb{Z}}\tilde{\Omega}_l$. Since $(L^*, \mathrm{dom}(L^*))$ is self-adjoint, 0 is either in the point spectrum or in the continuous spectrum. In case 0 is in the point spectrum, it is also contained in the point spectrum of $(J^*, \mathrm{dom}(J^*))$, so there exists a non-trivial function $t \mapsto v(t)$ such that $J_t^* v(t) = 0$ a.e. on Ω. By Theorem 3.3, however, the point 0 is not contained in the point spectrum of $(J_t^*, \mathrm{dom}(J_t^*))$ for any $t \in \Omega$, so $v(t) = 0$ a.e. and 0 belongs to the continuous spectrum. □

In order to make Theorem 4.6 more explicit we establish the corresponding spectral decomposition. Following the ideas of the proof of [23, Theorem XIII.86] we define

$$\mathcal{H}_n^+ = \left\{ v \in \int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt \,\middle|\, v(t) = f(t)\frac{\psi(q^n; t)}{N_{q^n}(t)} \text{ for some } f \in L^2(\Omega)\right\}, \quad n \in \mathbb{Z}_+, \tag{4.8}$$

and

$$\mathcal{H}_r^- = \left\{ v \in \int_\Omega^\oplus \ell^2(\mathbb{Z})\, dt \,\middle|\, v(t) = f(t)\frac{\psi(-q^r/t; t)}{N_{-q^r/t}(t)} \text{ for some } f \in L^2(\Omega)\right\}, \quad r \in \mathbb{Z}, \tag{4.9}$$

using the notation $N_\xi(t) = \|\psi(\xi;t)\|_{\ell^2(\mathbb{Z})}$ for ξ in the point spectrum of J_t^*. Then $\mathcal{H}_n^+$, $\mathcal{H}_r^-$ are mutually orthogonal closed subspaces of $\int_\Omega^\oplus \ell^2(\mathbb{Z})\,dt$ and, moreover,

$$\int_\Omega^\oplus \ell^2(\mathbb{Z})\,dt = \mathcal{H}^+ \oplus \mathcal{H}^- \quad \text{with } \mathcal{H}^+ = \bigoplus_{n=0}^\infty \mathcal{H}_n^+ \text{ and } \mathcal{H}^- = \bigoplus_{r=-\infty}^\infty \mathcal{H}_r^-.$$

Note that the subspaces $\mathcal{H}_l^\pm$ are contained in $\mathrm{dom}(J^*)$ and J^* preserves each of them. By $U_l^\pm\colon \mathcal{H}_l^\pm \to L^2(\Omega)$ we denote the unitary operator defined by $U_l^\pm v = f$ for $v \in \mathcal{H}_l^\pm$ of the form as in (4.8) or (4.9). It follows that $U_l^\pm$ intertwines J^* with multiplication by $\lambda_l^\pm$ on $L^2(\Omega)$, where $\lambda_l^+(t) = q^l$ and $\lambda_l^-(t) = -q^l/t$. We put $J_l^\pm = U_l^\pm J^*(U_l^\pm)^*$ so that $J_l^\pm f = \lambda_l^\pm f$ for all $f \in L^2(\Omega)$. In particular, it follows that $\ker(J^* - q^l) = \mathcal{H}_l^+$ so that $q^{\mathbb{Z}_+}$ is contained in the point spectrum of J^*, and each point of this form has infinite multiplicity.

For the case of negative eigenvalues we define $\tilde{\Omega}_l = \{-q^l/t \mid t \in \Omega\} \subseteq (-q^{l-1}, -q^l]$ for $l \in \mathbb{Z}$. Then $V_l : L^2(\Omega) \to L^2(\tilde{\Omega}_l)$ given by

$$(V_l f)(\lambda) = \frac{q^{l/2}}{|\lambda|} f(-q^l/\lambda), \qquad \lambda \in \tilde{\Omega}_l$$

is a unitary operator and its adjoint V_l^* is almost given by the same formula,

$$(V_l^* g)(t) = \frac{q^{l/2}}{t} g(-q^l/t), \qquad t \in \Omega.$$

By a straightforward calculation we see that

$$(V_l J_l^- V_l^* g)(\lambda) = \lambda\, g(\lambda), \qquad \lambda \in \tilde{\Omega}_l \tag{4.10}$$

for any $g \in L^2(\tilde{\Omega}_l)$. It thus follows that $\tilde{\Omega} = \cup_{l\in\mathbb{Z}} \tilde{\Omega}_l \subseteq (-\infty, 0]$ is contained in the continuous spectrum of J^*, and this part of the spectrum is simple. Using the notation $E(T|A)$ for the spectral projection corresponding to the Borel set $A \subset \mathbb{R}$ for a (possibly unbounded) self-adjoint operator T, we see that $E(V_l J_l^- V_l^* | A)$ is just multiplication by the characteristic function $\chi_{A\cap\tilde{\Omega}_l}$. Tracing the steps back it follows that

$$E(J^*|_{\mathcal{H}_l^-} | A) v(t) = \chi_{A\cap\tilde{\Omega}_l}(-q^l/t) v(t),$$

with the notation as in (4.8) and (4.9). By considering J^* restricted to $\mathcal{H}^-$, we see that $\sigma(J^*|_{\mathcal{H}^-}) = \overline{\cup_{l\in\mathbb{Z}} \tilde{\Omega}_l}$.

To obtain the spectral decomposition E of $(L^*, \mathrm{dom}(L^*))$ we use Proposition 4.4 and Theorem 4.6. The idea is to get the results from the spectral decomposition for J^* using the unitary isomorphism I. First we consider the spectral decomposition corresponding to the point spectrum $\sigma_p(L^*)$. It follows that L^* preserves $I^*\mathcal{H}^+$ and

$$\mathrm{ran}\big(E(\{q^n\}\big) = I^*\mathcal{H}_n^+ = \big\{\mathrm{Per}(f/\sqrt{w}) \cdot s_n \mid f \in L^2(\Omega)\big\},$$

where s_n is the orthonormal Stieltjes–Wigert polynomial of degree n. Note that by the functional equation (4.1), we have

$$\mathrm{Per}\big(f/\sqrt{w}\big)(x) = \frac{(Pf)(x)}{\sqrt{w(x)}}, \quad (Pf)(x) = \sum_{k=-\infty}^{\infty} \chi_{(q^{k+1},q^k]}(x)\, x^{k/2} q^{-k(k+1)/4} f(xq^{-k})$$

and $(Pf)(xq) = \sqrt{x}(Pf)(x)$. In particular, by taking any orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ of $L^2(\Omega)$ we obtain from the orthonormality of $t \mapsto f_j(t)\psi(q^n;t)/N_{q^n}(t)$ in $\mathcal{H}^+$ and the unitarity of I the orthogonality relations

$$\int_0^\infty \operatorname{Per}\big(f_i/\sqrt{w}\big)(x)\operatorname{Per}\big(f_j/\sqrt{w}\big)(x)s_n(x)s_m(x)w(x)\,dx$$
$$= \int_{\operatorname{supp}(\mu)} (Pf_i)(x)\,(Pf_j)(x)\,s_n(x)s_m(x)\,dx = \delta_{n,m}.\delta_{i,j}. \tag{4.11}$$

The special case $i = j$ tells us that the Stieltjes–Wigert polynomials are orthogonal with respect to any absolutely continuous measure whose density satisfies the functional equation (4.1). This result is also obtained in [10, Proposition 2.1].

To sum up, we denote by PPol $\subset L^2(\mu)$ the closure of the space of functions of the form $\sum f_n p_n \in L^2(\mu)$, with f_n a q-periodic function and p_n a polynomial. It follows that PPol $= I^*\mathcal{H}^+ \subset \operatorname{dom}(L^*)$ and $L^*|_{\mathrm{PPol}}$ is a bounded linear operator on PPol with spectrum $q^{\mathbb{Z}_+} \cup \{0\}$.

We now take a closer look at the spectral decomposition corresponding to the continuous spectrum of L^*. For any Borel set $A \subset (-q^{l-1}, -q^l]$ we have $E(A)I^*\mathcal{H}_r^- = \{0\}$ unless $r = l$. Since $E(A)F = I^*E(J^*|A)IF$ for $F \in L^2(\mu)$ with compact support, it thus follows that

$$E(J^*|A)(IF)(t) = \chi_{A\cap\tilde{\Omega}_l}(-q^l/t)\frac{\big\langle (IF)(t), \psi(-q^l/t;t)\big\rangle_{\ell^2(\mathbb{Z})}}{N_{-q^l/t}(t)}\frac{\psi(-q^l/t;t)}{N_{-q^l/t}(t)}.$$

Calculating I^* on $\mathcal{H}_l^-$ gives

$$I^*\left(t \mapsto f(t)\frac{\psi(-q^l/t;t)}{N_{-q^l/t}(t)}\right)(x)$$
$$= I^*\left(t \mapsto \frac{f(t)}{N_{-q^l/t}(t)}\sum_{k=-\infty}^{\infty} t^{k/2}q^{k(k+1)/4}\phi_{-q^l/t}(tq^k)e_k\right)(x)$$
$$= \sum_{k=-\infty}^{\infty}\chi_{(q^{k+1},q^k]}(x)\frac{f(xq^{-k})x^{k/2}}{N_{-q^{l+k}/x}(xq^{-k})}\frac{q^{-k(k+1)/4}}{\sqrt{w(x)}}\phi_{-q^{l+k}/x}(x),$$

so when f has the form

$$f(t) = \chi_{A\cap\tilde{\Omega}_l}(-q^l/t)\frac{\big\langle (IF)(t), \psi(-q^l/t;t)\big\rangle_{\ell^2(\mathbb{Z})}}{N_{-q^l/t}(t)},$$

we obtain for $G \in L^2(\mu)$ with compact support that

$$\langle E(A)F, G\rangle_{L^2(\mu)} = \int_0^\infty \big(I^*E(J^*|A)IF\big)(x)\,\overline{G(x)}\,w(x)\,dx$$
$$= \int_0^\infty \sum_{k=-\infty}^{\infty}\chi_{(q^{k+1},q^k]}(x)\frac{\chi_A(-q^{l+k}/x)}{N_{-q^{l+k}/x}(xq^{-k})^2}\,x^{k/2}q^{-k(k+1)/4}\phi_{-q^{l+k}/x}(x)$$
$$\times\Big\langle (IF)(xq^{-k}), \psi(-q^{l+k}/x; xq^{-k})\Big\rangle_{\ell^2(\mathbb{Z})}\overline{G(x)}\sqrt{w(x)}\,dx.$$

Expanding the inner product in the integrand, the integral can be written as

$$\sum_{j,k=-\infty}^{\infty} \int_{q^{k+1}}^{q^k} \chi_A(-q^{l+k}/x) \frac{\phi_{-q^{l+k}/x}(x)\phi_{-q^{l+k}/x}(xq^{j-k})}{N_{-q^{l+k}/x}(xq^{-k})^2} x^j q^{\binom{j+1}{2}-jk}$$
$$\times F(xq^{j-k})\overline{G(x)}w(x)\,dx$$
$$= \sum_{j,k=-\infty}^{\infty} (-1)^{j+k} q^{\binom{j+1}{2}\binom{k+1}{2}+l(j+k)} \int_A \frac{\phi_\lambda(-q^{l+j}/\lambda)\phi_\lambda(-q^{l+k}/\lambda)}{N_\lambda(-q^l/\lambda)^2\,\lambda^{j+k}}$$
$$\times F(-q^{l+j}/\lambda)\overline{G(-q^{l+k}/\lambda)}w(-q^l/\lambda)\frac{q^l}{\lambda^2}\,d\lambda$$
$$= \int_A \left(\sum_{j=-\infty}^{\infty} F(-q^{l+j}/\lambda)(-q^l/\lambda)^j q^{\binom{j+1}{2}} \phi_\lambda(-q^{l+j}/\lambda)\right)$$
$$\times \left(\sum_{k=-\infty}^{\infty} \overline{G(-q^{l+k}/\lambda)}(-q^l/\lambda)^k q^{\binom{k+1}{2}} \phi_\lambda(-q^{l+k}/\lambda)\right) \frac{q^l w(-q^l/\lambda)}{\lambda^2 N_\lambda(-q^l/\lambda)^2}\,d\lambda, \tag{4.12}$$

using the functional equation (4.1), switched to $\lambda = -q^{l+k}/x$. Note that

$$\sum_{j=-\infty}^{\infty} F(-q^{l+j}/\lambda)(-q^l/\lambda)^j q^{\binom{j+1}{2}} \phi_\lambda(-q^{l+j}/\lambda)$$
$$= \frac{(-\lambda)^l}{q^{\binom{l+1}{2}}} \sum_{j=-\infty}^{\infty} F(-q^j/\lambda)(-\lambda)^{-j} q^{\binom{j+1}{2}} \phi_\lambda(-q^j/\lambda)$$

and define

$$\big(\mathcal{F}F\big)(\lambda) = \sum_{j=-\infty}^{\infty} F(-q^j/\lambda)(-\lambda)^{-j} q^{\binom{j+1}{2}} \phi_\lambda(-q^j/\lambda). \tag{4.13}$$

By means of (4.13) we can write (4.12) as

$$\langle E(A)F, G\rangle_{L^2(\mu)} = \int_A \big(\mathcal{F}F\big)(\lambda)\overline{\big(\mathcal{F}G\big)(\lambda)}\lambda^{2l} q^{-l(l+1)} \frac{q^l w(-q^l/\lambda)}{\lambda^2 N_\lambda(-q^l/\lambda)^2}\,d\lambda$$
$$= \int_A \big(\mathcal{F}F\big)(\lambda)\overline{\big(\mathcal{F}G\big)(\lambda)}|\lambda|^l q^{-l(l+1)/2} \frac{w(-1/\lambda)}{N_\lambda(-q^l/\lambda)^2}\frac{d\lambda}{\lambda^2}, \tag{4.14}$$

using the functional equation (4.1) once more. Now define

$$v(\lambda) = \sum_{l=-\infty}^{\infty} \chi_{(-q^{l-1},-q^l]}(\lambda) \frac{|\lambda|^l q^{-l(l+1)/2}}{N_\lambda(-q^l/\lambda)^2} \tag{4.15}$$

and use (4.14) to obtain

$$\langle E(A)F, G\rangle_{L^2(\mu)} = \int_A \big(\mathcal{F}F\big)(\lambda)\overline{\big(\mathcal{F}G\big)(\lambda)}\, v(\lambda) w(-1/\lambda)\,\frac{d\lambda}{\lambda^2} \tag{4.16}$$

for an arbitrary Borel set $A \subset (-\infty, 0)$. It follows that the complex measure $\langle E(A)F, G\rangle_{L^2(\mu)}$ is absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$, and for any $F, G \in I^*\mathcal{H}^-$

we have

$$\langle F, G\rangle_{L^2(\mu)} = \int_{-\infty}^{0} \big(\mathcal{F}F\big)(\lambda)\overline{\big(\mathcal{F}G\big)(\lambda)}\, v(\lambda)w(-1/\lambda)\,\frac{d\lambda}{\lambda^2}. \tag{4.17}$$

Taking into account the discrete spectrum of L^* on the space PPol as well, we obtain the following Plancherel type theorem.

Theorem 4.7. *Consider an absolutely continuous positive measure μ on $(0,\infty)$ with density w satisfying the functional equation* (4.1). *Let $\Omega = (q,1] \cap \operatorname{supp}(\mu)$ and suppose that $\{f_i\}_{i=0}^{\infty}$ is an arbitrary fixed orthonormal basis of $L^2(\Omega)$. For all $F, G \in L^2(\mu)$, we have the Plancherel equality*

$$\int_0^{\infty} F(x)\overline{G(x)}w(x)\,dx = \sum_{i,n=0}^{\infty} F_{in}\overline{G_{in}} + \int_{-\infty}^{0} \big(\mathcal{F}F\big)(\lambda)\overline{\big(\mathcal{F}G\big)(\lambda)}\, v(\lambda)w(-1/\lambda)\,\frac{d\lambda}{\lambda^2},$$

where

$$F_{in} = \int_0^{\infty} F(x)\mathrm{Per}\big(f_i/\sqrt{w}\big)(x)s_n(x)w(x)\,dx$$

and $\mathcal{F}$, respectively v, are defined in (4.13) *and* (4.15).

We can rewrite the above result in terms of a corresponding transform. Consider the Hilbert space

$$\mathcal{K} = \ell^2\big(\mathbb{Z}_+ \times \mathbb{Z}_+\big) \oplus L^2\left((-\infty,0), v(\lambda)w(-1/\lambda)\frac{d\lambda}{\lambda^2}\right)$$

and define

$$(\mathcal{F}^*g)(x) = \sum_{i,n=0}^{\infty} g_{in}\mathrm{Per}\big(f_i/\sqrt{w}\big)(x)s_n(x) + \sum_{j=-\infty}^{\infty} g(-q^j/x)\phi_{-q^j/x}(x)v(-q^j/x), \quad x > 0 \tag{4.18}$$

for compactly supported functions $g \in \mathcal{K}$. If we consider $\mathcal{F}$ as defined in (4.13) as $\mathcal{F}:\ I^*\mathcal{H}^- \to L^2\Big((-\infty,0), v(\lambda)w(-1/\lambda)\frac{d\lambda}{\lambda^2}\Big)$ and extend it to an operator $\mathcal{F}:\ L^2(\mu) \to \mathcal{K}$ by defining $\mathcal{F}:\ I^*\mathcal{H}^+ \to \ell^2\big(\mathbb{Z}_+ \times \mathbb{Z}_+\big)$ by $\mathcal{F}F = \{F_{in}\}_{i,n\in\mathbb{Z}_+}$ with F_{in} as in Theorem 4.7, then we have the following result.

Corollary 4.8. *$\mathcal{F}:\ L^2(\mu) \to \mathcal{K}$ is a unitary isomorphism with adjoint given by* (4.18).

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